Classical and quantum position-dependent mass harmonic oscillators

S. Cruz y Cruz a,b,* , J. Negro a , L.M. Nieto a

a Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain
b Departamento de Física, Cinvestav, AP 14-740, México 07000 DF, Mexico

Received 24 April 2007; accepted 8 May 2007
Available online 17 May 2007
Communicated by P.R. Holland

Abstract

The position-dependent mass oscillator is studied from both, classical and quantum mechanical points of view, in order to discuss the ambiguity on the operator ordering of the kinetic term in the quantum framework. The results are illustrated by some examples of specific mass functions.

© 2007 Elsevier B.V. All rights reserved.
PACS: 03.65.Sq
Keywords: Position-dependent mass oscillators

1. Introduction

The interest in studying the Schrödinger equation with position-dependent mass (PDM) has been growing in the last fifty years due to its use in describing some physical phenomena as, e.g., the behavior of the charge carriers in semiconductor heterostructures, nuclear many body problems, and quantum dots physics, among some others [1–3]. The PDM concept is, by itself, a fundamental problem which is far from being completely understood. Some contributions have been developed from a theoretical point of view [4,5] and other approaches to find solutions or to generate exactly solvable problems have been also carried out [6–10].

An important point in this framework concerns the ambiguity of the ordering operator in the kinetic term of the Hamiltonian. This question has been addressed in many references and from different points of view, see for instance [1,4]. In this Letter we want to consider this problem through the PDM oscillator by taking into account classical as well as quantum mechanical aspects. We will see that there is a special ordering in the quantum mechanical frame which is the closest to the classical picture. The organization of the Letter is as follows. In Section 2 we will study the classical PDM harmonic oscillator paying special attention to the canonical transformation with the usual constant mass (CM) oscillator. Section 3 is devoted to choosing a Hamiltonian for the PDM quantum mechanical oscillator. It will be shown that there is a one parameter family decomposing this Hamiltonian into a kinetic plus a potential term. Just one element of this family gives an expression for the potential which coincide with the classical one in the previous section. The point-wise transformation to the CM quantum mechanical oscillator is also considered to make clear the similarities with the classical counterpart. These properties will be illustrated by means of three examples of the mass function, for which the behavior near a singularity or the boundaries at infinity is studied. We end this work with some concluding remarks.

2. Classical position-dependent mass harmonic oscillator

2.1. General considerations

Consider a classical PDM system described by the Lagrangian [11–13]
\[ \mathcal{L} = \frac{1}{2} m(x) \dot{x}^2 - \mathcal{V}(x), \]  
(1)

with the corresponding equation of motion
\[ m(x) \ddot{x} + \frac{1}{2} \frac{d}{dx} m(x) \dot{x}^2 + \mathcal{V}'(x) = 0. \]  
(2)

This is a generalization of Newton’s second law of motion for systems with PDM. With the momentum \( p \) defined in the usual way
\[ p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m(x) \dot{x}, \]
the Legendre transformation gives the corresponding Hamiltonian \([11–14]\)
\[ \mathcal{H} = \frac{p^2}{2m(x)} + \mathcal{V}(x). \]  
(3)

In this Letter we will choose the potential \( \mathcal{V}(x) \) by making use of some algebraic properties of the constant mass harmonic oscillator. It is in this sense that we set \( \mathcal{H} \) as the Hamiltonian of the PDM harmonic oscillator.

Let us factorize \( \mathcal{H} \) in terms of the functions
\[ A^{\pm} = \mp i \frac{p}{\sqrt{2m(x)}} + \mathcal{W}(x), \]  
(4)

with \( \mathcal{W}(x) \) a function to be determined. Then
\[ \mathcal{H} = A^+ A^- - A^- A^+ = \frac{p^2}{2m(x)} + \mathcal{W}^2(x), \]  
(5)

and the potential \( \mathcal{V}(x) \) must then satisfy
\[ \mathcal{V}(x) = \mathcal{W}^2(x). \]  
(6)

We now demand \( A^{\pm} \) to hold the same Poisson bracket as the CM classical harmonic oscillator, i.e.,
\[ \{ A^-, A^+ \} = \frac{\partial A^-}{\partial p} \frac{\partial A^+}{\partial x} - \frac{\partial A^-}{\partial x} \frac{\partial A^+}{\partial p} = i, \]
and use this condition to determine \( \mathcal{W} \) (up to an additive constant which can be absorbed by making a coordinate displacement):
\[ \mathcal{W}(x) = \frac{1}{2} \int \sqrt{2m(x)} \, dx. \]  
(7)

Hence, the PDM harmonic oscillator potential, in terms of the mass function has the form
\[ \mathcal{V}(x) = \frac{1}{2} \left( \int \sqrt{m(x)} \, dx \right)^2. \]  
(8)

It is not difficult to show that
\[ \{ \mathcal{H}, A^{\pm} \} = \pm i A^{\pm}, \]  
(9)

and that one can construct non-autonomous constants of motion of the form
\[ Q^{\pm} = A^{\pm} e^{\mp it} = \left( \mp i \frac{p}{\sqrt{2m(x)}} + \mathcal{W}(x) \right) e^{\mp it}, \]  
(10)

holding
\[ Q^+ Q^- = \mathcal{E}, \]
where \( \mathcal{E} \) is the total energy of the system. This allows us to determine the trajectories \( x(t) \), \( p(t) \), in the phase space, in terms of \( \mathcal{E} \) and a phase displacement \( \phi \) which is fixed by the initial conditions of the problem:
\[ x(t) = \mathcal{W}^{-1} \left( \sqrt{\mathcal{E}} \cos(t + \phi) \right), \]  
(11)
\[ p(t) = -\sqrt{2\mathcal{E}} \sqrt{m(x(t))} \sin(t + \phi). \]  
(12)

To illustrate how this formalism works, in particular, for some limiting situations, different forms of the mass function were chosen. We will concentrate here on the details of the solutions; the reasons why these functions were chosen will be explained later. Notice that, even when \( m(x) \) has an unusual form, the behavior of the dynamical variables is quite regular. In these examples \( m_0, \lambda \) are positive constant parameters, and the CM case may be recovered in the limit \( \lambda \to 0 \).

(a) Mass without singularities
As a first example we will consider the mass function
\[ m_1(x) = \frac{m_0}{1 + (\lambda x)^2}. \]  
(13)

It is a bounded function, defined in the whole real line \( (\mathcal{D}(m_1) = \mathbb{R}) \) taking its maximum value, \( m_0 \), at \( x = 0 \) and vanishing as \( |x| \to \infty \). The potential and the phase trajectories are
\[ \mathcal{V}(x) = \frac{m_0}{2\lambda^2} \arcsin^2 \lambda x, \]  
\[ x(t) = \frac{1}{\lambda} \sin \left[ \sqrt{\frac{2\mathcal{E}^2}{m_0}} \cos(t + \phi) \right], \]  
\[ p(t) = -\sqrt{2\mathcal{E}m_0} \frac{\sin(t + \phi)}{\cosh \left[ \sqrt{\frac{2\mathcal{E}^2}{m_0}} \cos(t + \phi) \right]}. \]  
(14)

(b) Mass with one singularity
Next, let us consider the case
\[ m_2(x) = \frac{m_0}{(1 + \lambda x)^2}. \]  
(15)

This function is defined in \( \mathcal{D}(m_2) = (-1/\lambda, \infty) \); it becomes divergent near \( x = -1/\lambda \) and vanishes as \( x \to +\infty \). In this case, the potential and the phase trajectories are
\[ \mathcal{V}(x) = \frac{m_0}{2\lambda^2} \ln^2 (1 + \lambda x), \]  
\[ x(t) = \frac{1}{\lambda} \left[ e^{\sqrt{\frac{2\mathcal{E}^2}{m_0}} \cos(t + \phi)} - 1 \right], \]  
\[ p(t) = -\sqrt{2\mathcal{E}m_0} e^{-\sqrt{\frac{2\mathcal{E}^2}{m_0}} \cos(t + \phi)} \sin(t + \phi). \]  
(16)

(c) Mass with two singularities
Finally, consider the function
\[ m_3(x) = \frac{m_0}{(1 - (\lambda x)^2)^2}. \]  
(17)

The motion is now confined to the region \( \mathcal{D}(m_3) = (-1/\lambda, 1/\lambda) \). The mass of our system rapidly grows near \( x = \pm 1/\lambda \) and
reaches its minimum at $x = 0$. The potential and phase trajectories in this case are given by

$$V(x) = \frac{m_0}{2\lambda^2} \arctanh^2 \lambda x,$$

$$x(t) = \frac{1}{\lambda} \tanh \left[ \sqrt{2E/\lambda^2 m_0} \cos(t + \phi) \right],$$

$$p(t) = -\sqrt{2E m_0} \cosh^2 \left[ \sqrt{2E/\lambda m_0} \cos(t + \phi) \right] \sin(t + \phi). \quad (18)$$

In all cases, it is not difficult to verify that, in the limit $\lambda \to 0$, the potential $V(x)$, as well as the phase space coordinates $x(t)$, $p(t)$ reduce to that of the CM harmonic oscillator:

$$V(x) = \frac{1}{2} m_0 \lambda^2, \quad x(t) = \sqrt{2E/\lambda m_0} \cos(t + \phi),$$

$$p(t) = -\sqrt{2E m_0} \sin(t + \phi). \quad (19)$$

Fig. 1 shows the trajectories in the phase space for $m_1(x)$, $m_2(x)$, $m_3(x)$, with $m_0 = \lambda = 1$, phase displacement $\phi = 0$, and different values of $E$. In case (a) the curves are just soft deformations of the phase trajectories of the CM oscillator (concentric circumferences centered at the origin); the position and momentum variables taking, in principle, arbitrary values. However, in cases (b), (c), it is evident the presence of singularities in the mass function, restricting the motion of the particle to a confined zone determined by $\mathcal{D}(m)$ in each case. It can be noted also that the values of the momentum $p$ rapidly grow as $x$ tends to its limiting values, being this fact more visible for greater values of $E$.

2.2. Correspondence with the constant mass harmonic oscillator

Suppose we now perform the following transformation of phase space coordinates:

$$\mathcal{X}(x) = \sqrt{2} \mathcal{V}(x), \quad \mathcal{P}(x, p) = \sqrt{\frac{p}{m(x)}}. \quad (20)$$

It is not difficult to prove that $\{\mathcal{X}, \mathcal{P}\} = 1$, so this is actually a canonical transformation. A direct inspection shows that the Hamiltonian (3) takes the form

$$\mathcal{H} = \frac{1}{2}(\mathcal{P}^2 + \mathcal{X}^2), \quad (21)$$

and that Eqs. (11)–(12) are reduced to the phase trajectories (19) of the constant mass harmonic oscillator $(m_0 = 1)$. This shows that there is a close correspondence between the PDM and the CM oscillators: one can be transformed into the other by means of the canonical transformation (20) [13].

So far, the choice of the mass has been quite arbitrary; however, for some functional forms of $m(x)$ the canonical transformation (20) may not map $\mathcal{D}(m)$ onto the whole real line, as it is required if $\mathcal{X}(x)$ represents the position of a CM oscillator. The following conditions must be considered:

1. Behavior near a singularity. Let us suppose, e.g., that $m(x) \sim 1/(x - x_0)^q$ near $x = x_0$, with $q > 0$; then $\mathcal{X}(x) \sim \int (x - x_0)^{-q/2} dx$ contains a singularity in $x = x_0$ for $q \geq 2$. This implies that $\mathcal{X}(x)$ will take values in all $\mathbb{R}$ as $x$ ranges along $(x_0, +\infty)$. For $q < 2$, however, $\mathcal{X}(x)$ would be restricted to the half-real line and would not represent the position of the conventional harmonic oscillator.

2. Asymptotic behavior. Now let us suppose that $m(x) \sim 1/x^q$ as $x \to +\infty$. A similar reasoning shows that, in order the standard oscillator coordinate $\mathcal{X}(x)$ ranges over the whole real line, we must take $q \leq 2$.

Therefore, $m_1(x), m_2(x), m_3(x)$ are not arbitrary choices of the mass function, they were chosen in such a way that they fulfill the above requirements near the boundaries.

3. Quantum position-dependent mass oscillator

3.1. General considerations

In the framework of quantum mechanics, the dynamical variables $(x, p)$, in the coordinate representation, are the linear operators $(x, p = -i\hbar d/dx)$, then, a generic PDM Hamiltonian like (taking $\hbar = 1$)

$$H = -\frac{1}{2} m^a \frac{d}{dx} m^b \frac{d}{dx} m^a + V(x), \quad a + b = -\frac{1}{2}. \quad (22)$$
present an ordering ambiguity of \( p \) and \( m(x) \) expressed by the parameters \( a, b \). In order to construct the PDM harmonic oscillator Hamiltonian in quantum mechanics, we will proceed in a similar way to the classical case. We must define the creation–annihilation operators \( A^\pm \), keeping in mind again the ordering uncertainty in the differential term. Let, e.g., \( A^+_a \) be of the form

\[
A^+_a = \frac{1}{\sqrt{2}} m^a \frac{d}{dx} m^a + W_a(x),
\]

(23)

and \( W_a(x) \) a function of the coordinate operator known as the superpotential. Then, we can write two harmonic oscillator Hamiltonians \( H^\pm_a \) as

\[
H^+_a = A^+_a A^-_a = -\frac{1}{2} m^a \frac{d}{dx} m^a \frac{d}{dx} m^a + V^+_a(x) = T^+_a + V^+_a,
\]

(24)

\[
H^-_a = A^-_a A^+_a = -\frac{1}{2} m^b \frac{d}{dx} m^b \frac{d}{dx} m^b + V^-_a(x) = T^-_a + V^-_a,
\]

(25)

where \( T^\pm_a \) are the kinetic terms and the corresponding potentials \( V^\pm_a \) are given in terms of the superpotentials \( W_a(x) \) as

\[
V^+_a(x) = -\left( \frac{W_a}{\sqrt{2m}} \right)' - 4a W_a \left( \frac{1}{\sqrt{2m}} \right) + W^2,
\]

(27)

and

\[
V^-_a = -\left( \frac{W_a}{\sqrt{2m}} \right)' - 4a W_a \left( \frac{1}{\sqrt{2m}} \right) + W^2 + 2W^2
\]

\[- 4a + 1 \left( \frac{1}{\sqrt{2m}} \right)' .\]

(28)

The form of \( W_a(x) \) is fixed (up to an additive constant which we set equal to zero by means of a displacement) by demanding \( A^\pm \) to hold the Heisenberg algebra, i.e., \([A^-_a, A^+_a] = 1\):

\[
W_a(x) = \frac{1}{2} \int \sqrt{2m} \frac{d}{dx} + \frac{4a + 1}{2} \left( \frac{1}{\sqrt{2m}} \right)'.
\]

(29)

Thus, the potentials \( V^\pm_a(x) \) for the PDM harmonic oscillator, in terms of the mass are given by

\[
V^\pm_a(x) = \frac{1}{2} \left( \int \sqrt{m} \frac{dx}{\sqrt{dx}} \right)^2 - \frac{4a + 1}{4} \left( \frac{1}{\sqrt{2m}} \right) + \left( \frac{1}{\sqrt{2m}} \right) + \left( \frac{1}{\sqrt{2m}} \right).\]

(30)

The parameter \( a \) in these expressions distinguishes different forms of \( V^\pm_a \) for different choices of \( T^\pm_a \). However, the Hamiltonians \( H^\pm_a \) will be the same for any value of \( a \), and this parameter only labels a particular ordering in the kinetic term. This result was already obtained in [10] for \( a = 0 \) (compare also [8]).

3.2. Correspondence with the classical case

The arbitrariness in the quantum PDM harmonic oscillator potential is not present in the classical case, where the potential and kinetic terms are well determined. Notice that, when \( a = -1/4 \), the form of \( V^\pm_a \) is as close as possible to the classical potential \( V(x) \) given by (8):

\[
V^\pm(x) = V^\pm_{-1/4}(x) = \frac{1}{2} \left( \int \sqrt{m} \frac{dx}{\sqrt{dx}} \right)^2 + \frac{1}{2}.
\]

(31)

Fig. 2 illustrates well this situation. It shows, respectively, the oscillator potential for the masses \( m_1(x) \), \( m_2(x) \), \( m_3(x) \), for \( m_0 = \lambda = 1 \) and different values of \( a \), compared with that of the classical case. In all of them, it is not difficult to notice that, as \( a \) moves away from \(-1/4\), the shape of \( V^\pm_a \) becomes significantly different from the classical oscillator potential \( V(x) \). This fact suggests that the most suitable ordering in the kinetic term is that given by this value of \( a \). We can then define the quantum PDM harmonic oscillator Hamiltonian as

\[
H^\pm = A^\pm A^{\mp} = \frac{1}{2} \int \frac{d}{\sqrt{m}} \frac{d}{\sqrt{dx}} \frac{d}{\sqrt{m}} + V^\pm(x),
\]

(32)

with creation–annihilation operators

\[
A^\pm = A^\pm_{-1/4} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{m}} \frac{dx}{\sqrt{dx}} \frac{1}{\sqrt{m}} + \int \sqrt{m} \frac{dx}{\sqrt{dx}} \right).
\]

(33)

As in the case of constant mass, \( H^+ = H^- - 1 \), and the spectral values and eigenfunctions of \( H^+ \) are given respectively by

\[
E_n = n, \quad n = 0, 1, 2, \ldots ,
\]

(34)
and
\[ \psi_n(x) = \frac{1}{\sqrt{\mathcal{N}}} (A^+)^n \psi_0(x), \] (35)
with \( \psi_0(x) \) the (normalized) state which is annihilated by \( A^- \) and has eigenvalue \( E_0 = 0 \).

### 3.3. Correspondence with the constant mass harmonic oscillator

In the quantum frame we can also make a correspondence between the CM and PDM quantum oscillators. In the constant mass case (with \( m_0 = 1 \)) the Hamiltonian reads
\[ H^+ = -\frac{1}{2} \frac{d^2}{dy^2} + \frac{1}{2} y^2 - \frac{1}{2} = a^+ a, \] (36)
the creation–annihilation operators given by
\[ a^\pm = \frac{1}{\sqrt{2}} \left( \mp \frac{d}{dy} + y \right), \] (37)
and have eigenfunctions which can be written in the form
\[ \phi_n(y) = \frac{1}{\sqrt{\mathcal{N}}} (a^+)^n \phi_0(y) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} e^{-y^2/2} H_n(y), \] (38)
where \( H_n(y) = (-)^n e^{y^2/2} \frac{d^n}{dy^n} e^{-y^2} \) are the Hermite polynomials, corresponding to the eigenvalues \( E_n = n \). It turns out that the functions \( \psi_n(x) \) in (35) can be written in terms of the eigenfunctions of the CM oscillator, by means of the transformation [15,16]:
\[ y = y(x), \quad \psi_n(x) = f(y(x)) \phi_n(y(x)), \] (39)
where
\[ y(x) = \int \sqrt{m(x)} dx, \quad f(y(x)) = \frac{1}{\sqrt{m(x)}}. \] (40)

The Hamiltonian \( H^+ \) can then be written as:
\[ H^+ = A^+ A^- = -\frac{1}{2} \frac{1}{\sqrt{m(x)}} \frac{d^2}{dy(x)^2} \frac{1}{\sqrt{m(x)}} + \frac{1}{2} y(x)^2 - \frac{1}{2}, \] (41)
with
\[ A^\pm = \frac{1}{\sqrt{2}} \left( \pm \frac{1}{\sqrt{m(x)}} \frac{d}{dy(x)} \frac{1}{\sqrt{m(x)}} + y(x) \right) \] (42)
and it will have square integrable eigenfunctions given by (39):
\[ \int_{\mathcal{D}(m)} |\psi_n(x)|^2 dx = \int_{-\infty}^{\infty} |f(y) \phi_n(y)|^2 \frac{1}{(f(y))^2} dy \]
\[ = \int_{-\infty}^{\infty} |\phi_n(y)|^2 dy. \]

Notice that the transformation (40) corresponds to the canonical transformation connecting the classical CM and PDM oscillators. In order to achieve the correct transformation of domains, we are assuming here that \( m(x) \) satisfy the same requirements near the boundaries as in Section 2.2. In this case the boundary behavior of the wave functions are also preserved.

As illustrative examples we present the cases for the mass functions \( m_1(x), m_2(x), m_3(x) \) considered before, since they allow to construct wave functions with the proper behavior near the boundaries (we have set \( m_0 = 1 \) for the sake of simplicity):

**(a) Mass with no singularities**
\[ m(x) = \frac{1}{1 + (\lambda x)^2}, \quad y(x) = \frac{1}{\lambda} \arcsinh \lambda x, \]
\[ H^+ = -\frac{1}{2} \sqrt{1 + (\lambda x)^2} \frac{d}{dx} \sqrt{1 + (\lambda x)^2} \frac{d}{dx} \sqrt{1 + (\lambda x)^2} \]
\[ + \frac{1}{2\lambda^2} \arcsinh^2 \lambda x - \frac{1}{2}, \] (43)
\[ \psi_n(x) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} \sqrt{1 + (\lambda x)^2} e^{-\frac{1}{2\lambda^2} \arcsinh^2 \lambda x} \]
\[ \times H_n \left( \frac{1}{\lambda} \arcsinh \lambda x \right). \] (44)

**(b) Mass with one singularity**
\[ m(x) = \frac{1}{1 + (\lambda x)^2}, \quad y(x) = \frac{1}{\lambda} \ln(1 + \lambda x), \]
\[ H^+ = -\frac{1}{2} \sqrt{1 + \lambda x} \frac{d}{dx} \left( 1 + \lambda x \right) \frac{d}{dx} \sqrt{1 + \lambda x} \]
\[ + \frac{1}{2\lambda^2} \ln^2(1 + \lambda x) - \frac{1}{2}, \] (45)
\[ \psi_n(x) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} \sqrt{1 + \lambda x} e^{-\frac{1}{2\lambda^2} \ln^2(1 + \lambda x)} \]
\[ \times H_n \left( \frac{1}{\lambda} \ln(1 + \lambda x) \right). \] (46)

**(c) Mass with two singularities**
\[ m(x) = \frac{1}{1 - (\lambda x)^2}, \quad y(x) = \frac{1}{\lambda} \arctan \lambda x \]
\[ H^+ = -\frac{1}{2} \sqrt{1 - (\lambda x)^2} \frac{d}{dx} \left( 1 - (\lambda x)^2 \right) \frac{d}{dx} \sqrt{1 - (\lambda x)^2} \]
\[ + \frac{1}{2\lambda^2} \arctan^2 \lambda x - \frac{1}{2}, \] (47)
\[ \psi_n(x) = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} \sqrt{1 - (\lambda x)^2} e^{-\frac{1}{2\lambda^2} \arctan^2 \lambda x} \]
\[ \times H_n \left( \frac{1}{\lambda} \arctan \lambda x \right). \] (48)

Fig. 3 shows the potential \( V^+(x) \) and the first three eigenfunctions of \( H^+ \) for \( \lambda = 1 \). Notice the strong similarity of case (a) with the CM harmonic oscillator. Even cases (b), (c) show some similarities despite the visible deformation of \( V^+(x) \) and the corresponding wave functions due to the presence of singularities.
3.4. Classical and quantum densities

We finally present a comparison between the classical presence density \( \rho_c(x, E) \), which is proportional to the inverse of the velocity of the system, and the quantum probability density \( \rho_q(x, n) = |\psi_n(x)|^2 \).

In the classical case the motion of the system is confined to a closed interval of the real line defined by the turning points \( x_1, x_2 \), where the total energy equals the potential energy. The velocity is \( v = \sqrt{2(E - V(x))/m} \), so the presence density reads

\[
\rho_c(x, E) = \frac{1}{\pi} \sqrt{\frac{m(x)}{2(E - V(x))}},
\]

and fulfills

\[
\int_{x_1}^{x_2} \rho_c(x, E) \, dx = 1.
\]

The presence density should fit the quantum probability density as \( n \) grows. Fig. 4 shows the graphics of the probability density contrasted with that of the presence density for the classical and quantum states with energy \( E_n = \mathcal{E} = 4 \), in the cases \( m_1(x), m_2(x), m_3(x) \). We see, in our examples, that density \( \rho_c \) fits reasonably well the quantum density. There are, however, differences in the specific quantum wave behavior. Also, one can appreciate that, as \( m \to 0 \), the quantum density \( \rho_q \) becomes negligible. In regions where \( m(x) \) increases, both, \( \rho_c \), and \( \rho_q \), also grow, specially near \( x = 0 \) for case (a), and near the singular points for cases (b), (c).

4. Concluding remarks

We have considered different decompositions of the Hamiltonian of the PDM oscillator in quantum mechanics. Such decompositions correspond to different reordering of the kinetic term. Then, we have argued that just one of this decomposition gives rise to a potential term that is the same as the classical PDM oscillator where there are no ambiguity in the ordering. With this specific potential term, the correspondence with the CM oscillator in the classical and quantum mechanical frames is much closer. We have shown also how to deal with singularities and boundary conditions of the mass functions. A quite important result is the big change that the singular cases cause in the range of the PDM coordinate. In this way, for instance, we have been able to define a PDM oscillator in a bounded domain introducing masses with singularities such as \( m_2 \) and \( m_3 \). Finally, we have shown that the PDM \( m(x) \) determines, in classical as well as in quantum frames, a change of the density \( \rho(x) \). We can say that the effect of a mass increase is similar to that of a potential well, while the vanishing of \( m(x) \) reproduces a potential barrier.

Acknowledgements

This work is partially supported by the Spanish MEC (MTM2005-09183 and FIS2005-03989) and Junta de Castilla y León (Excellence project VA013C05). S.C.yC. thanks the
members of the Theoretical Physics Department, Universidad de Valladolid (Spain) for kind hospitality. The support of Conacyt (grant SEP-2004-C01–47200/A-1 and projects 50766 and 49253-F) and COTEPABE, COFAA and SIP (IPN, Mexico), is acknowledged.

References