MEASUREMENT IN CONTROL AND DISCRIMINATION OF ENTANGLED PAIRS UNDER SELF-DISTORTION

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Quantum correlations and entanglement are fundamental resources for quantum information and quantum communication processes. Developments in these fields normally assume stable resources, not susceptible of distortion. That is not always the case, Heisenberg interactions between qubits can produce distortion on entangled pairs generated for engineering purposes (e. g. quantum computation or quantum cryptography). The presence of parasite magnetic fields modifies the expected properties and behavior for which the pair was intended. Quantum measurement and control help to discriminate the original state in order to correct it or reconstruct it using some procedures which do not alter their quantum nature. Different kinds of quantum entangled pairs driven by a Heisenberg Hamiltonian with an additional inhomogeneous magnetic field become distorted. They can be reconstructed by adding an external magnetic field with fidelity close to one. In addition, each state can be efficiently discriminated. Combining both processes, first reconstruction without discrimination and after discrimination with adequate non-local measurements, it is possible a) improve the discrimination, and b) reprepare faithfully the original state. The complete process gives fidelities better than 0.9. Some results about a class of equivalence for the required measurements are found, allowing to select the experimentally most adequate.

Keywords: Quantum control; Heisenberg model; quantum discrimination; quantum measurements

1. Introduction

Quantum information processing has been a field of increasing activity in diverse areas such as information theory, control, engineering and measuring\(^1\). Physical elements of quantum information applications are bounded by physical imperfections like decoherence or self distortion by interactions between their parts. Additional procedures have been introduced to correct these deviations. Quantum control deals with this kind of problems by measuring, analyzing and providing feedback to the system in order to implement those procedures given that its alteration upon measurement is not completely quantifiable. Quantum control was propelled by notable works on control in nanoscale systems\(^2\), in quantum feedback control\(^3,4\) and in quantum systems under continuous feedback\(^5,6\). Some specific procedures of quantum control are based on exploiting the system properties in order to drive
it without the use of projective or weak measurements\textsuperscript{7–9}. Processes to introduce quantum control have been recently developed in order to discriminate states by taking measurements and using feedback on single qubits\textsuperscript{10–13} and on entangled qubits\textsuperscript{14}.

Quantum correlations and entanglement are the basis for quantum computation and quantum communication. Normally, the stability of these items is assumed, but in almost any physical system, that is not always true. Some recent experimental work shows how to produce entangled spin qubits in quantum dots and electron gases\textsuperscript{15,16}, where identification and control are essential for later applications, given that input variations in the initial configuration of the entangled pair are present. In relation to this, Ref.\textsuperscript{17} has presented some schemes for discrimination and reparation to prevent the distortion which could emerge due to external and non-controllable interactions enabled by parasite fields which can distort the original state. In addition, some applications require precise knowledge of the produced state, so quantum measurement is necessary to characterize it without altering it.

In this paper, we show that both, discrimination and reparation, are efficiently made by a previous reconstruction of the distorted state without discrimination, in agreement with the scenario and the set of prescriptions given in Ref.\textsuperscript{17}. Discrimination is optimal for a class of generic equivalent measurements (ranging from local to non-local measurements) and its performing gives an efficient reparation. By generic, we mean independent from distortion parameters. The paper is organized as follows: Section\textsuperscript{2} explains reconstruction process related to results in Ref.\textsuperscript{17}. Section\textsuperscript{3} outlines the discrimination process and the kind of measurements involved. In Section\textsuperscript{4}, equivalent measurements for discrimination are discussed. In Section\textsuperscript{5}, we depict the reparation of the original states after discrimination.

2. Distortion Control in Heisenberg Interaction for Bipartite Qubits

We deal with the problem depicted in Fig.\textsuperscript{1}. A system generates known entangled states with the same probability, one state at a time out of two possible non-equivalent states. The user does not know which specific state is produced and in addition, some kind of distortion is introduced by magnetic interaction between their parts before he has access to it. For quantum engineering purposes, this state is required to be identified and to be reconstructed into the original state in the best possible way. This is a simplification of the process presented inRefs.\textsuperscript{15 and 16} and the control focus differs from that presented in Ref.\textsuperscript{17}. Here we are interested in both operations at once, instead only one. Thus, we deal with an analogous system as shown in Ref.\textsuperscript{17}, in which, an entangled pair in one of the two possible orthogonal states is generated:

\[
|\beta_1\rangle = |\beta_{00}\rangle
\]
\[
|\beta_2\rangle = \sin \theta |\beta_{01}\rangle - \cos \theta |\beta_{10}\rangle,
\]
where $|\beta_{ij}\rangle$ (for $i,j \in \{0,1\}$) are the standard Bell states. As in Ref. 17, $\theta$ is a parameter which provides a monotone distinction between the states. The trace distance for them is $\delta(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} |\rho_1 - \rho_2| = \sin^2 \theta$, where $\rho_1 = |\beta_1\rangle \langle \beta_1|$ and $\rho_2 = |\beta_2\rangle \langle \beta_2|$. The last state, $|\beta_2\rangle$, goes from a state similar to $|\beta_1\rangle$ when $\theta = 0$ to a very different one when $\theta = \pi/2$. Note that initially, both states are maximally entangled.

Immediately after, a self-interaction ruled by the Heisenberg Hamiltonian with a possible inhomogeneous magnetic field in some direction begins between the pair of particles:

$$H = -J \sigma_1 \cdot \sigma_2 + B_1 \sigma_{1z} + B_2 \sigma_{2z}. \quad (2)$$

In agreement with Refs. 14 and 17, after some time $t$, those states become distorted:

$$|\beta'_1\rangle = e^{itJ} (\cos(b_+ t') |\beta_{00}\rangle - i \sin(b_+ t') |\beta_{10}\rangle)$$

$$|\beta'_2\rangle = (e^{-ij'} (2i j \sin t' + \cos t') \sin \theta |\beta_{01}\rangle - e^{itJ} \cos \theta \cos(b_+ t') |\beta_{10}\rangle) + \{i e^{-ij'} \cos \theta \sin(b_+ t') |\beta_{00}\rangle - ib_+ e^{-ij'} \sin \theta \sin t' |\beta_{11}\rangle\}, \quad (3)$$

where $b_+ = (B_1 + B_2)/R$, $b_- = (B_1 - B_2)/R$, $j = J/R$, $R = \sqrt{(B_1 - B_2)^2 + 4J^2}$ and $t' = Rt$. In the following we drop the prime in the time variable. This interaction is trace distance preserving: $\delta(\rho'_1, \rho'_2) = \sin^2 \theta$. The first state in (3) always remains maximally entangled but not the second one; some entanglement is lost due to the

![Fig. 1. Graphic description of the complete process of discrimination and repreparation including the Heisenberg distortion. The system generates one out of two possible non-equivalent bipartite entangled states. Then, during time $t$, the magnetic interaction introduces an internal distortion. The user applies for some time $T$ a reconstruction procedure, still under the Heisenberg interaction. In order to reduce the possibility of error in the discrimination, then, a suitable measurement is applied to discriminate the original state. Finally a repreparation process is used to recover the original state.](image-url)
distortion. Ref. 17 shows two different control procedures to reprepare the original state. Here we select the first on-site control procedure which is more accurate. In the following we refer to it as a reconstruction process instead of a repreparation process as in that work, because we need to differentiate it from the repreparation process followed a later discrimination. By applying an extra homogeneous magnetic field during time $T$ after $t$, we obtain the complete evolution operator in terms of two operators:

$$U(t + T) = U_{b_+} + \delta b_+ (T) U_{b_+} (t), \quad (4')$$

the first one drives the distortion and the second one drives the reconstruction (in a different average field $b_+ \rightarrow b_+ + \delta b_+$, with $b_-$ unchanged), in agreement with Ref. 14. This last operation has the control parameters:

$$T = n\pi - t$$

$$\delta b_+ = \frac{\pi (2m - n(b_+ - 2j + 1))}{T}$$

$$\frac{s}{2n} = Q(j);$$

with: $n, m, s \in \mathbb{Z}$, \quad (5')

where $Q(j)$ is a rational approximation to $j$ (with $2n$ as denominator). With a suitable selection of $n$ and $s$, we can have $Q(j)$ as close to $j$ as we want (in the case $j \in \mathbb{Q}$, $j = Q(j)$ is always possible). Thus, we obtain a quasi evolution loop (up to unitary factors) $U(t + T) = I'$, with $I'$ the diagonal matrix: $I' = \text{diag}(1, 1, 1, e^{4in\pi \delta})$, where $\delta = j - Q(j)$. The reconstructed states become:

$$|\beta''_1\rangle = (1 + ie^{2in\pi \delta} \sin 2n\pi \delta) |\beta_{00}\rangle - ie^{2in\pi \delta} \sin 2n\pi \delta |\beta_{10}\rangle,$$

$$|\beta''_2\rangle = \sin \theta |\beta_{01}\rangle - e^{2in\pi \delta} \cos \theta \cos 2n\pi \delta |\beta_{10}\rangle + ie^{2in\pi \delta} \cos \theta \sin 2n\pi \delta |\beta_{00}\rangle. \quad (6')$$

Selection of a suitable $n$ brings $\cos 4n\pi \delta$ as close to 1 as needed. In this sense, some restrictions related to the nature and knowledge of $j$ are noted in Ref. 17. In this way, the reconstruction’s average fidelity becomes:

$$F_N = \frac{1}{2} \left( |\langle \beta_1 | \beta''_1 \rangle |^2 + |\langle \beta_2 | \beta''_2 \rangle |^2 \right) =$$

$$= 1 - \frac{1}{2} \sin^2 2n\pi \delta (1 + \cos^2 \theta). \quad (7')$$

The reason for a previous reconstruction is clear: to take advantage of the same properties of the magnetic field which generates the distortion in order to make a more feasible discrimination process than that obtained in Ref. 17. Note that if
no further discrimination and repreparation is made, then \( F_N \) is the Do-nothing process fidelity.

3. Measurement Problem for Discrimination

3.1. General problem of measurement for discrimination

The general problem of measurement for discrimination is given by a set of measurement operators \( \{ M_i \vert i = 1, \ldots, m \} \):

\[
\sum_{i=1}^{m} E_i = \sum_{i=1}^{m} M_i^† M_i = 1. \tag{8}
\]

From this set of operators we identify two subsets which correspond to each qubit to be identified:

\[
\mathcal{E}_i + \mathcal{E}_f = 1
\]

\[
\mathcal{E}_k = \sum_{i \in \{i \vert f(i) = k\}} E_i
\]

with: \( k = 1, 2 \), \( \tag{9} \)

where \( f(i) = k \) assigns each measurement operator \( E_i \) with some \( \mathcal{E}_k \) in agreement with some predefined criteria. Then, the average fidelity is:

\[
\bar{F} = \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} P_{H_k}^{(j)} \text{Tr}(\bar{\rho}_j \rho_k), \tag{10}
\]

where \( \rho_k, \bar{\rho}_k \) and \( \bar{\rho}_k \) are the density matrices for desired, pre-measured and repreared states respectively. \( P_{H_k}^{(j)} \) are the Helmstrom probabilities and their complements:

\[
P_{H_k} = P_{H_k}^{(k)} = \text{Tr}(\bar{\rho}_k \mathcal{E}_k)
\]

\[
1 - P_{H_k} = P_{H_k}^{(\neq k)} = \text{Tr}(\bar{\rho}_k \mathcal{E}_j), \tag{11}
\]

for each qubit \( k = 1, 2 \).

3.2. Basis measurement transformations and fidelity invariance subgroup

In order to discriminate the states presented in the last section, we need to make some measurement in order to decide if the premeasured state (here, the distorted plus the reconstructed state) was \( |\beta_1\rangle \) or \( |\beta_2\rangle \). The goal of this subsection is to study
some unitary transformation which can be made on the state prior to measurement, defining in this way an alternative measurement (Fig. 2):

\[
M_i' = M_i U \Rightarrow E'_i = M_i'^\dagger M_i' = U^\dagger E_i U \\
\Rightarrow \mathcal{E}_k' = U^\dagger \mathcal{E}_k U.
\] (12)

We are especially interested on those measurements with the form:

\[
\mathcal{E}_k = \sum_{i,j \in s} |\phi_{ij}\rangle \langle \phi_{ij}|,
\] (13)

where \(s = \{(i, j) | 2 - \delta_{ij} = k; i, j = 0, 1\}\) and \(\mathcal{B} = \{|\phi_{ij}\rangle | i, j = 0, 1\}\) is a complete orthogonal bipartite basis, not necessarily separable in general. The interest in them is because they are the experimentally simpler measurements. In particular, through this work we will use the three measurement bases (with element definitions as in (13) in that order):

\[
\mathcal{B}_C = \{|00\rangle, |01\rangle, |11\rangle, |10\rangle\}
\]

\[
\mathcal{B}_B = \{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}
\] (14)

\[
\mathcal{B}_R = \{|\rho_{00}\rangle = |\beta_{00}\rangle, \\
\rho_{01}\rangle = \sin \theta |\beta_{01}\rangle - \cos \theta |\beta_{10}\rangle, \\
\rho_{10}\rangle = -\cos \theta |\beta_{01}\rangle - \sin \theta |\beta_{10}\rangle, \\
\rho_{11}\rangle = |\beta_{11}\rangle\}
\] (15)

Note that there is an implicit correspondence between the elements trough of the three basis, which will be important below. We will be interested in the unitary transformations \(U\) which leave the fidelity invariant. Following equations (10) and (11) we get:

![Quantum circuit containing a unitary transformation U preceding the measurement and followed by an additional transformation U' (used for repreparation, for example). The whole effect is to define a new measurement \(M_i' = M_i U\). Note that the measurement device includes both qubits indicating a possible non-local measurement.](image)
Measurement in control and discrimination of entangled pairs under self-distortion

\[ P_{H_k}^{(j)} = \langle \beta_k' | \mathcal{E}_j | \beta_k' \rangle = \langle \beta_k' | U \mathcal{E}_j U^\dagger | \beta_k' \rangle = \langle \beta_k' | \mathcal{E}_j | \beta_k' \rangle \]

so that the strong condition is \([\mathcal{E}_j, U] = 0\), it means that \(\{\mathcal{E}_j\}\) is invariant. Imposing the last commuting condition on a general \(U\) expressed in an arbitrary basis \(\mathcal{B}\), we obtain by direct calculation (note that \(|\mathcal{E}_1, U| = 0\) implies \(|\mathcal{E}_2, U| = 0\):

\[
U = \begin{pmatrix}
    a_{1,1} & 0 & 0 & b_{1,1} \\
    0 & a_{2,1} & b_{2,1} & 0 \\
    0 & b_{2,2} & a_{2,2} & 0 \\
    b_{1,2} & 0 & 0 & a_{1,2}
\end{pmatrix}
\]

with:

\[
a_{i,j}^2 + b_{i,j}^2 = 1; \quad i, j = 1, 2
\]

\[
\sum_{k, l} a_{i,k} b_{i,l} = 0; \quad i = 1, 2.
\]

By direct calculation we can show that these special transformations are a subgroup of the unitary transformations. One special case is the transformation (which will be useful for our discussion):

\[
U_{CB} = C_3^I \mathcal{H} C_3^I
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix}
    1 & 0 & 0 & 1 \\
    0 & 1 & 1 & 0 \\
    0 & 1 & -1 & 0 \\
    1 & 0 & 0 & -1
\end{pmatrix},
\]

(expressed in the computational basis) where \(\mathcal{H}\) is the Hadamard gate operator and \(C_3^I\) is the controlled-not gate. It transforms the set:

\[
\mathcal{M}_C = \{ \mathcal{E}_{C_1} = |00\rangle \langle 00| + |11\rangle \langle 11|, \mathcal{E}_{C_2} = |01\rangle \langle 01| + |10\rangle \langle 10| \},
\]

into the equivalent set:

\[
\mathcal{M}_B = \{ \mathcal{E}_{B_1} = |\beta_{00}\rangle \langle \beta_{00}| + |\beta_{10}\rangle \langle \beta_{10}|, \mathcal{E}_{B_2} = |\beta_{01}\rangle \langle \beta_{01}| + |\beta_{11}\rangle \langle \beta_{11}| \}.
\]
Another case is the transformation:

\[
U_{BR} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin \theta & -\cos \theta & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

expressed in the \(B_R\) basis, which transforms:

\[
\mathcal{M}_{B'} = \{\mathcal{E}_{B'_1} = |\beta_{00}\rangle \langle \beta_{00}| + |\beta_{11}\rangle \langle \beta_{11}|, \\
\mathcal{E}_{B'_2} = |\beta_{01}\rangle \langle \beta_{01}| + |\beta_{10}\rangle \langle \beta_{10}|\},
\]

in:

\[
\mathcal{M}_{R} = \{\mathcal{E}_{R_1} = |\rho_{00}\rangle \langle \rho_{00}| + |\rho_{11}\rangle \langle \rho_{11}|, \\
\mathcal{E}_{R_2} = |\rho_{01}\rangle \langle \rho_{01}| + |\rho_{10}\rangle \langle \rho_{10}|\},
\]

note that \(U_{CB}^\dagger = U_{CB}\) and \(U_{BR}^\dagger = U_{BR}\).

3.3. Measurement implementation

Now, we will describe how to implement the last set of measurements. For the measurements in the computational basis \(B_C\), Fig. 3a shows the single measuring of spin in \(z\) direction on each particle. Fig. 3b shows the prior application of transformation \(U_{CB}\) followed by the measurement in the computational basis \(B_C\) in the same order of identification which was mentioned regarding formula (14). In some sense, this transformation is equivalent to the measurement in the nonlocal \(B_B\) basis. Note that an additional application of the inverse transformation \(U_{CB}\) on the measured state (not included in the figure) gives the correct measured state in the Bell basis, nevertheless as a further process of reconstruction follows, this inverse transformation is skipped now. Finally, Fig. 3c shows the same procedure as in Fig. 3b but using ancilla qubits to make the measurements by entangling them with the qubits actually being discriminated (in the sense of Neumark’s theorem\(^{19}\)). This last measurement process is included here only as reference of a cleaner process which does not touch the qubits of interest.

In agreement with the last section, the use of measuring sets \(M_C\) or \(M_B\) are completely equivalent for the fidelity of the process (assuming that an additional reconstruction on the measured state into the original state could be made for both cases). This sets are included in this work to argue that the intuitive basis, \(M_C\), used for discrimination in Ref. 17, is equivalent to use nonlocal measurements based on \(M_B\), because it does not give improving results for the fidelity.

Similarly, if the measuring set \(M_{B'}\) is selected for the discrimination by using the last equivalent procedure with nonlocal measurement on Bell basis. It will be
equivalent, in the sense of fidelity invariance, to the process using $\mathcal{M}_R$ (but not using $\mathcal{M}_C$ or $\mathcal{M}_B$), in the order of identification given in (14). In this sense, an additional procedure based on the $U_{BR}$ transformation to measure in that basis with this set is not necessary. The importance on the implementation of these measurements

Fig. 3. Quantum circuits containing different measurement implementations. a) Basic measurement in the computational basis, b) transformation for non local measurement in the Bell basis and c) implementation of Bell non local measurements with ancilla qubits.
will be explained in the following sections. Fig. 4 depicts the relations between the basis of measurement, the measurement operators and the fidelity invariance transformations discussed before.

4. Optimum Practical Measurements and Fidelities

In the present section, we will discuss some useful measurement for our control problem. At this point, we have the reconstructed states (6) with qubits set far from each other so Heisenberg interaction stops, but other local or non-local operations can still be applied to them. The actual states form suggests the use of non-local measurements in the Bell basis. Nevertheless, it’s well know\textsuperscript{20} that the optimal POVM to discriminate both states is:

\[
\mathcal{E}_k = M_k^\dagger M_k
\]

\[
M_{k \neq k'} = \frac{1 - |\beta_{k'}^{\alpha'}\rangle \langle \beta_{k'}^{\alpha'}|}{1 + |\langle \beta_k^{\alpha'}| \beta_{k'}^{\alpha'}\rangle|^2}, \quad k, k' = 1, 2
\]

\[
\mathcal{E}_{\text{inc}} = 1 - \mathcal{E}_1 - \mathcal{E}_2, \quad \tag{24}\]

where $\mathcal{E}_{\text{inc}}$ is stated for measurements which are inconclusive for discrimination (note that $\langle \beta_{k'}^{\alpha'}| \mathcal{E}_{\text{inc}} | \beta_{k'}^{\alpha'}\rangle = |\langle \beta_k^{\alpha'}| \beta_{k'}^{\alpha'}\rangle|$ for $k' = 1, 2$). Nevertheless, as in this situation both states remain orthogonal, this is equivalent to take for $M_k$:

\[
M_{\text{opt}} = |\beta_k^{\alpha'}\rangle \langle \beta_k^{\alpha'}|, \quad k = 1, 2. \quad \tag{25}\]

Unfortunately this measurement basis is not always experimentally practical because it depends on the reconstruction parameters $n$ and $\delta$.

4.1. Bell basis measurements and optimum measurement

As a first approximation to the problem of finding a practical basis of measurement, we propose a kind of measurements of the form:

![Fig. 4. Fidelity invariance transformations relating different measurement basis.](image)
\begin{equation}
M = \sum_{i,j=0}^{1} \alpha_{ij} |\beta_{ij}\rangle \langle \beta_{ij}|.
\end{equation}

Solving the optimization problem for \(\alpha_{ij}\) parameters in order to maximize \(P = \langle \beta'_{k'} | M M | \beta'_{k'} \rangle\) for each \(k' = 1, 2\), a direct calculation gives the critical \(P\)'s for \(k' = 1\):

\begin{align}
M_{1a} &= |\beta_{00}\rangle \langle \beta_{00}| \Rightarrow P_{1a} = \cos^2 2n\pi \delta \\
M_{1b} &= |\beta_{10}\rangle \langle \beta_{10}| \Rightarrow P_{1b} = \sin^2 2n\pi \delta,
\end{align}

and for \(k' = 2\):

\begin{align}
M_{2a} &= |\beta_{00}\rangle \langle \beta_{00}| \Rightarrow P_{2a} = \sin^2 2n\pi \delta \cos^2 \theta \\
M_{2b} &= |\beta_{10}\rangle \langle \beta_{10}| \Rightarrow P_{2b} = \cos^2 2n\pi \delta \cos^2 \theta \\
M_{2c} &= |\beta_{11}\rangle \langle \beta_{11}| \Rightarrow P_{2c} = \sin^2 \theta.
\end{align}

As is expected, they depend on \(n\) and \(\delta\) and there is a conflict with both sets of measurements because, depending on those values, the assignment could be made to either \(|\beta''_{k}\rangle\) or \(|\beta''_{k'}\rangle\) discrimination. Here, we will assume that we are in the \(\cos 4\pi n \delta = 1\) case, so it is more convenient to assign \(|\beta_{00}\rangle \langle \beta_{00}|\) for the discrimination of \(|\beta''_{k}\rangle\) and \(|\beta_{10}\rangle \langle \beta_{10}|\) for the discrimination of \(|\beta''_{k'}\rangle\). Note that \(|\beta_{11}\rangle \langle \beta_{11}|\) is not only inconclusive for both cases but \(\langle \beta_{11}| \beta''_{k'} \rangle = 0\), so its measurement operator can be included in \(\mathcal{E}_{inc}\), or instead in \(\mathcal{E}_1\) or \(\mathcal{E}_2\) without effect. For simplicity in the notation used in Section 2, we include it in \(\mathcal{E}_1\). It means that this is equivalent to use \(\mathcal{M}_R\) as measurement operators for discrimination.

Nevertheless, the use of \(\mathcal{B}_B\) in (26) is not general; there is an argument about the practicality to use the last measurement operators. Extending the set (25) with two other orthogonal operators \(|\beta''_{k}\rangle \langle \beta''_{k'}|\) for \(k = 3, 4\). This problem can be solved by obtaining \(1 - M_{op}^{\dagger} M_{op} - M_{op}^{\dagger} M_{op},\) and then diagonalizing and calculating its eigenvectors. This subspace is degenerate so the selection is not unique. A convenient selection is constructed with:

\begin{align}
|\beta''_{3}\rangle &= -e^{-4ni\pi \delta} \cos \theta |\beta_{01}\rangle \\
&\quad + ie^{-4ni\pi \delta} \sin 2n\pi \delta \sin \theta |\beta_{00}\rangle \\
&\quad - e^{-2ni\pi \delta} \cos 2n\pi \delta \sin \theta |\beta_{10}\rangle, \\
|\beta''_{4}\rangle &= |\beta_{11}\rangle.
\end{align}

One interesting aspect is that in the \(\cos 4\pi n \delta = 1\) case, this set of measurements reduces exactly to \(\mathcal{M}_R\) which is equivalent to \(\mathcal{M}_B\).
4.2. Fidelities

We can analyze the Helstrom probabilities by calculating $|\langle \phi_{ij} | \beta''_k \rangle |^2$ for each basis element of $|\phi_{ij}\rangle$ and for each reconstructed state $|\beta''_k\rangle$ (Table 1). With them, it is easy to calculate the Helstrom probabilities for each measurement operator set $\mathcal{M}$ and the final fidelity. In the case of C and B, we get:

$$F_C = F_B = 1 - \frac{1}{2} \cos^2 \theta. \quad (30)$$

And for B' and R:

$$F_{B'} = F_R = 1 - \frac{1}{2} \sin^2 2n\pi\delta(1 + \cos^2 \theta) = F_N. \quad (31)$$

It was expected that no fidelity better than $F_N$ will be obtained. The only advantage is that in this process we have discriminated the states without additional cost (assuming that we are able to reprepare the states faithfully after discrimination). The last fidelities are compared in Fig. 5. It is noticeable that near the $\cos 4\pi n\delta = 1$ case and independently of $\theta$, fidelity is practically one. Far away from this case, the effects of the specific value of magnetic field and $\theta$ mentioned in Ref. 17 are present through the imperfect reconstruction after the distortion. Note specially that near the case $\theta \approx 0$, $\mathcal{M}_C$ and $\mathcal{M}_B$ measuring operators become a better option than $\mathcal{M}_{B'}$ or $\mathcal{M}_R$.

4.3. Experimental limitations

In Ref. 17 was mentioned that the unknowledge of $j$ can restrict the control effectiveness. The problem is that the term $n\delta$ in our expressions is not limited to go onto zero in a controllable way in order to make that $F \rightarrow 1$. While better $Q(j)$ will be to go $j$, normally $n$ is greater. By writing $j$ (assuming $N$ integer digits) as a sum of powers of ten: $j = \sum_{i=-N}^{\infty} a_i10^{-i}$, we can take a

| Basis | $|\beta''_0\rangle$ | $|\beta''_1\rangle$ |
|-------|------------------|------------------|
| $|00\rangle$ | $\frac{1}{2}$ | $\frac{1}{2} \cos^2 \theta$ |
| $|01\rangle$ | 0 | $\frac{1}{2} \sin^2 \theta$ |
| $|10\rangle$ | 0 | $\frac{1}{2} \sin^2 \theta$ |
| $|11\rangle$ | $\frac{1}{2}$ | $\frac{1}{2} \cos^2 \theta$ |

| $|\beta_{00}\rangle$ | $\cos^2 2n\pi\delta$ | $\cos^2 \theta \sin^2 2n\pi\delta$ |
| $|\beta_{01}\rangle$ | 0 | $\sin^2 \theta$ |
| $|\beta_{10}\rangle$ | $\sin^2 2n\pi\delta$ | $\cos^2 \theta \cos^2 2n\pi\delta$ |
| $|\beta_{11}\rangle$ | 0 | 0 |

| $|\beta_{00}\rangle$ | $\cos^2 2n\pi\delta$ | $\cos^2 \theta \sin^2 2n\pi\delta$ |
| $|\beta_{01}\rangle$ | $\cos^2 \theta \sin^2 2n\pi\delta$ | $\cos^2 \theta \cos^2 2n\pi\delta(1 + \sin^2 \theta) + \sin^4 \theta$ |
| $|\beta_{10}\rangle$ | $\sin^2 \theta \sin^2 2n\pi\delta$ | $\cos^2 \theta \sin^2 \theta \sin^2 2n\pi\delta$ |
| $|\beta_{11}\rangle$ | 0 | 0 |
rational approximation of order \( k \) only the first \( k \) decimal digits in \( j \), we get
\[
n \delta = n(j - \sum_{i=-k}^{k} a_i 10^{-i}) = 10^k \sum_{i=k+1}^{\infty} a_i 10^{-i} \approx (a_{k+1} + 1)/10 < 1.
\]
But, actually this is not a very useful bound because we assume that \( n = 10^5 \). Nevertheless, if \( Q(j) = s/2n \) in (5) is reducible, then the last bound can be much lower. If \( j \) is a number with an arbitrary number of digits, but our actual knowledge of it is only to the \( k \)-digit, we can still use a worse rational approximation \( Q(j) \) in order for \( n \delta \) becomes lower by seeking the adequate \( s \) and \( n \) which make \( Q(j) \) reducible. To illustrate this, we seek numerically the best values of \( Q(j) \) up to \( k = 5 \) digits for each \( j \in (0, 1] \) (note that \( j > 1 \) gives the same results for \( \sin^2 2\pi n \delta \) in (31)). Results are shown in Fig. 6. The graphic shows the best value of \( 1 - F_{\text{max}} \) (when \( \theta = 0 \)) versus \( j \) that is possible to find with up to \( k = 5 \) digits of approximation for \( Q(j) \). Each point depicts that solution and the whole graphic suggest the density of cases for each \( 1 - F_{\text{max}} \) value. This density is clearly lower for greater values of \( 1 - F_{\text{max}} \). The cumulative distribution function, \( \Omega \), corresponding to each \( 1 - F_{\text{max}} \) value is also shown. Note that around of 90% of cases have up of 0.8 in fidelity.

5. Final Repreparation Process

In this section we will analyze the final process of repreparation following the measurement and discrimination procedures studied before. In all cases we will assume that the measurement is made in the \( |ij \rangle \) basis, following the appropriate \( U \) transformation to induce an equivalent non-local measurement as was studied in the previous sections.

![Diagram](image)

Fig. 5. Comparison between \( F_C = F_B \) (black) and \( F_{B'} = F_R \) (chessboard layer) fidelities.
5.1. Basic gates for repreparation

In the present section we will present different quantum computational gates which serve to reprepare the state emerging from measurement (which is always possible, see Ref. 21 for example). In addition to $C^1_2$ gate (which we do not discuss here because is ample studied, see for example Bergman et al$^{22}$ and Wu and Zhang$^{23}$), the repreparation process from the measured states induced by $M_{tr}$ depicted in Section 3 will require the local gate $U_\theta = I \cos \theta - iZ \sin \theta$ which can be physically generated by the qubit interaction with a $z$-magnetic field $B_z$ during $\tau$ time:

$$U_{B_z, \tau} = \begin{pmatrix} e^{-iB_z \tau} & 0 \\ 0 & e^{iB_z \tau} \end{pmatrix}, \quad (32)$$

so $U_\theta = U_{B_z, \tau}$ with $B_z \tau = \theta$. An additional necessary gate is the phase gate $S$, which can be obtained from this last operation by selecting $B_z \tau = \frac{4p+1}{4} \pi, p \in \mathbb{Z}$ and up to a unitary factor:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad (33)$$

Hadamard gate can be generated up to a unitary factor through gates similar to $U_{B_z, \tau}$, by using magnetic fields in other directions:

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![Graphical representation](image.png)

**Fig. 6.** Best values for $1 - F_{\text{max}}$ for each $j \in (0, 1]$ by selecting the best $Q(j)$ with up to $k = 5$ digits which minimizes $\eta \delta$. Below, the corresponding cumulative distribution function, $\Omega$. 

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\[ U_{B_\sigma \tau} = I \cos B_\sigma \tau - i \Sigma \sin B_\sigma \tau, \quad (34) \]

with \( \sigma = x, y, z; \Sigma = X, Y, Z \) respectively. With this, the Hadamard gate becomes (up to a unitary factor):

\[ \mathcal{H} = U_{B_\sigma \tau} U_{B_\sigma' \tau'}, \quad (35) \]

with \( B_\sigma \tau = \frac{2p+1}{2} \pi, \quad B_\sigma' \tau' = \frac{4q+1}{4} \pi \) and \( p, q \in \mathbb{Z} \). Clearly, the operators \( X, Y \) and \( Z \) can be obtained (up to unitary factors) from (32, 34) by an adequate selection of \( B_i \tau = \frac{2p+1}{2} \pi \) with \( i = x, y, z \) respectively, and \( p \in \mathbb{Z} \).

### 5.2. Operations for the final repreparation

Based on the knowledge of the original state (with sufficient certainty), after of discrimination we can apply a set of transformations in order to recover the original state, in agreement with the fidelity (30). The central aspects here is to note that \( \mathcal{H} U_{\nu} \mathcal{H} |j\rangle = \cos \theta |j\rangle - i \sin \theta |j \oplus 1\rangle \), where \( j = 0, 1, \oplus \) is the module two sum; and \( C_1^2 \mathcal{H} C_2^1 \) is the transformation between the computational basis and the Bell basis depicted in Section 3. With that, one can easily reprepare the original state into the discriminated state by the set of measurements \( \mathcal{M}_B \) in (22), beginning with each measured state, \( |i_j\rangle \) obtained through the scheme depicted in the Fig. 3b (or Fig. 3c):

\[ |\beta_{j+1}\rangle = U_{j+1,i_j} |i_j\rangle. \quad (36) \]

Table 2 shows (up to unitary factors), the precise transformations obtained from basic local and non-local transformations in the last subsection.

### 6. Conclusions

The results of this work improve the control introduced in Ref. 14 by applying a general reconstruction before discrimination. After this, an additional process of repreparation gives an almost perfect recovering of the original state depending on.

| \( |i_j\rangle \) | \( U_{j+1,i_j} \) | \( |\beta_{j+1}\rangle \) |
|-----------------|----------------------|----------------------|
| \( |00\rangle \) | \( (C_1^2 \mathcal{H}_1 C_2^1) \) | \( |\beta_1\rangle \) |
| \( |01\rangle \) | \( (C_1^2 \mathcal{H}_1 C_2^1)(Y_1 S_1)(\mathcal{H}_1 U_{\beta_1} \mathcal{H}_1) \) | \( |\beta_2\rangle \) |
| \( |11\rangle \) | \( (C_2^1 \mathcal{H}_1 C_2^1)(Y_1 S_1 X_1)(\mathcal{H}_1 U_{\beta_1} \mathcal{H}_1) \) | \( |\beta_2\rangle \) |
upon the precise knowledge of the $j$ coupling constant. Notwithstanding the sensitivity on $j$ of this process, it is not a conditioning aspect for a successful control. The use of an equivalent measurement basis makes the use of appropriate non-local measurements experimentally easier, in order to discriminate adequately the state after reconstruction. Some of the details about the spatial management of pair of qubits are disregarded. Future work should be directed towards improving the control dependence on $j$ and the control of spatial position, using by example ion traps as in Ref. 8. This is an central aspect in the whole pair control process.

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References