ON VARIATIONAL AND EFFECTIVE PROPERTIES OF AN 1-D NONLINEAR COMPOSITE

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Introduction
A comparison between asymptotic homogenization and variational techniques, applied to one-dimensional nonlinear heterogeneous medium, is presented in order to predict different estimates for the effective energy.

Nonlinear Composites
We will develop approximations within the context of nonlinear elasticity. The constitutive equation in one dimension is

\[ \sigma = \frac{dw(y)}{dy} \]  

where \( w \) is the energy density, \( \sigma \) is the stress and \( y \) is the strain. The basic field equations are

\[ d\sigma + g(x) = \frac{dy}{dx}. \]  

For nonlinear media studied here \( w(y) \) is convex. We assume that the composite is 1-periodic. Let \( \varepsilon = 1/N \) a “small” geometric parameter and \( N \) the number of periodic cells on the segment of interest with length \( L \). The energy density for the composite depends on position \( x \) inside the periodic cell and has the form

\[ w\left( y, \frac{x}{\varepsilon} \right) = \sum_{r=1}^{n} w_{r}(y) f_{r}\left( \frac{x}{\varepsilon} \right) \]  

where \( f_{r}(x/\varepsilon) \) is the characteristic function of the region occupied by phase \( r \); it takes the value 1 in that phase and zero otherwise.

Asymptotic Homogenization Method (AHM)
By applying the Asymptotic Homogenization Method (AHM), as in Bahkvalov and Panasenko¹, an effective (homogenized) constitutive law for this composite is obtained as an asymptotic approximation when the size of the periodic cell is small as compared with the size of the composite, \( \varepsilon \ll L \). From Eqn. 1 and Eqn. 3 we have

\[ \sigma(y, \xi) = \sum_{r=1}^{n} \sigma_{r}(y) f_{r}(\xi); \quad \xi = \frac{x}{\varepsilon}. \]  

Here \( \xi = x/\varepsilon \) is a “fast” variable used at the level of the small, periodic cell, while \( x \) is used at the level of the large composite. Then the field Eqn. 2 becomes

\[ \frac{\partial}{\partial x} + \frac{1}{\varepsilon} \frac{\partial}{\partial \xi} \sigma\left( \frac{du}{dx}, \xi \right) = g(x, \xi). \]  

We seek an asymptotic solution of Eqn. 5 in the form

\[ u^{(v)} = v_{0}(x) + \varepsilon u_{1}(x, \xi) + \varepsilon^{2} u_{2}(x, \xi) + \ldots. \]  

Here the \( u_{i} \) are 1-periodic in \( \xi \). Replacing Eqn. 6 in Eqn. 5, expanding in powers of \( \varepsilon \) we obtain the homogenized equation

\[ d\sigma + g(x) = \frac{dv_{0}}{dx}. \]  

with

\[ g(x) = \int_{0}^{x} g(x, \xi) d\xi, \]  

\[ \sigma\left( \frac{dv_{0}}{dx} \right) = \sigma\left( x, \frac{dv_{0}}{dx} + \frac{\partial u}{\partial \xi} \right). \]  

Here \( \sigma \) does not depend on \( \xi \). Eqn. 9 is the homogenized (effective) constitutive law.

Variational Bounds
Another approach for homogenizing a composite is to obtain variational bounds on the effective properties ². The minimum energy principle states that the effective energy of the composite is given by

\[ \bar{w}(y) = \inf_{\Omega} \int_{\Omega} w(y(x), x) dx, \]  

\[ K = \{ y : y \text{ is } \Omega\text{-periodic, } y = du/dx, u = y \cdot x \}. \]
where $\Omega$ is the periodic cell, with unitary length. This variational principle leads to the elementary upper bound in the effective energy

$$w(y) \leq \overline{w}(y)$$

with $\overline{w}(y) = \int_\Omega w(y, x) \, dx$  \hspace{1cm} (11)

and (in a similar way based on the corresponding dual principle) the elementary lower bound

$$\widetilde{w}(y) \geq \left( \overline{w}^* \right) (y)$$

with $f^*(\sigma) = \sup_y \left\{ \sigma \cdot y - \overline{f}(y) \right\}$  \hspace{1cm} (12)

Improved bounds can be obtained by using a known comparison material.\textsuperscript{2, 3}

**Example**

Consider a two-phase composite with power-law energy $w_1(y) = \frac{1}{n} E_1 y^n, \quad w_2(y) = \frac{1}{n} E_2 y^n$

$$\sigma_1(y) = E_1 y^{n-1}, \quad \sigma_2(y) = E_2 y^{n-1}.$$  \hspace{1cm} (13)

In this case the AHM constitutive law (Eqn. 8 and 9) becomes

$$\hat{\sigma}_{AHM} = \hat{E} y^{n-1} \quad \hat{E}_{AHM} = \left( c_1 E_1^{\frac{1}{1-n}} + c_2 E_2^{\frac{1}{1-n}} \right)^{1-n}$$  \hspace{1cm} (14)

where $c_r$ is the volume fraction of phase $r$, $\sum c_r = 1$.

With $n = 2$ (linear case) $\hat{E}_{AHM}$ is the harmonic mean. With other values of $n$, $\hat{E}_{AHM}$ is the generalized Hölder Mean of order $1/(1-n)$. The corresponding effective energy is

$$\overline{W}_{AHM} = \frac{1}{n} \hat{E}_{AHM} y^n$$  \hspace{1cm} (15)

The stress against strain for a fraction $c_1 = 0.7$ and an exponent $n = 4$ are shown in fig. 1. Notice the geometrical relationship $\frac{\overline{Y}_2 Y}{\overline{Y}_1 Y} = c_1/c_2$.

On the other hand, the elementary upper bound (Eqn. 11) becomes:

$$\overline{W}_{EUB} = c_1 w_1 + c_2 w_2.$$  \hspace{1cm} (16)

The elementary lower bound is obtained from Eqn. 12. In this example (both phases with power-law energy with the same exponent) the elementary lower bound coincides with AHM effective energy, i.e.

$$\overline{W}_{ELB} \equiv \overline{W}_{AHM} \quad \text{(for power-law energy)}.$$  \hspace{1cm} (17)

The energy approximations as a function of the fraction $c_1$ are shown in Fig. 2.

**Concluding remarks**

Two different approaches for the homogenization of composite materials have been presented and compared. Future work\textsuperscript{4} includes the combination of both methods in order to improve bounds on effective properties.

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**References**